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Theory of resistor networks: the two-point resistance

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Abstract

The resistance between two arbitrary nodes in a resistor network is obtained in terms of the eigenvalues and eigenfunctions of the Laplacian matrix associated with the network. Explicit formulae for two-point resistances are deduced for regular lattices in one, two and three dimensions under various boundary conditions including that of a Möbius strip and a Klein bottle. The emphasis is on lattices of finite sizes. We also deduce summation and product identities which can be used to analyse large-size expansions in two and higher dimensions.

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1. Introduction

A classic problem in electric circuit theory studied by numerous authors over many years is the computation of the resistance between two nodes in a resistor network (for a list of relevant references up to 2000 see, e.g., [1]). Besides being a central problem in electric circuit theory, the computation of resistances is also relevant to a wide range of problems ranging from random walks (see [2] and [3]¹, and discussions below), the theory of harmonic functions [4], first-passage processes [5], to lattice Green's functions [6]. The connection with these problems originates from the fact that electrical potentials on a grid are governed by the same difference equations as those occurring in the other problems. For this reason, the resistance problem is often studied from the point of view of solving the difference equations, which is most conveniently carried out for infinite networks. In the case of Green's function approach, for example, past efforts [1, 7] have been focused mainly on infinite lattices. Little attention has been paid to *finite* networks, even though the latter are those occurring in real life. In this paper, we take up this problem and present a general formulation for computing two-point resistances in finite networks. Particularly, we show that known results for infinite networks are recovered by taking the infinite-size limit. In later papers we plan to study effects of lattice defects and carry out finite-size analyses.

¹ I am indebted to S Redner for calling my attention to this reference.

The study of electric networks was formulated by Kirchhoff [8] more than 150 years ago as an instance of a linear analysis. Our starting point is along the same line by considering the Laplacian matrix associated with a network. The Laplacian is a matrix whose off-diagonal entries are the conductances connecting pairs of nodes. Just as in graph theory where everything about a graph is described by its adjacency matrix (whose elements is 1 if two vertices are connected and 0 otherwise), everything about an electric network is described by its Laplacian. Indeed, in section 2 below we shall derive an expression for the two-point resistance between two arbitrary nodes in terms of the eigenvalues and eigenvectors of the Laplacian matrix². In the following sections our formulation is applied to networks of one-dimensional and two-dimensional lattices under various boundary conditions including those embedded on a Möbius strip and a Klein bottle, and lattices in higher spatial dimensions. We also deduce summation and product identities which can be used to reduce the computational labour, and analyse large-size expansions in two and higher dimensions.

Let \mathcal{L} be a resistor network consisting of \mathcal{N} nodes numbered $i = 1, 2, \dots, \mathcal{N}$. Let $r_{ij} = r_{ji}$ be the resistance of the resistor connecting nodes i and j . Hence, the conductance is

$$c_{ij} = r_{ij}^{-1} = c_{ji}$$

so that $c_{ij} = 0$ (as in an adjacency matrix) if there is no resistor connecting i and j .

Denote the electric potential at the i th node by V_i and the net current flowing *into* the network at the i th node by I_i (which is zero if the i th node is not connected to the external world). Since there exist no sinks or sources of current including the external world, we have the constraint

$$\sum_{i=1}^{\mathcal{N}} I_i = 0. \quad (1)$$

The Kirchhoff law states

$$\sum_{j=1}^{\mathcal{N}'} c_{ij}(V_i - V_j) = I_i \quad i = 1, 2, \dots, \mathcal{N} \quad (2)$$

where the prime denotes the omission of the term $j = i$. Explicitly, equation (2) reads

$$\mathbf{L}\vec{V} = \vec{I} \quad (3)$$

where

$$\mathbf{L} = \begin{pmatrix} c_1 & -c_{12} & \dots & -c_{1\mathcal{N}} \\ -c_{21} & c_2 & \dots & -c_{2\mathcal{N}} \\ \vdots & \vdots & \ddots & \vdots \\ -c_{\mathcal{N}1} & -c_{\mathcal{N}2} & \dots & c_{\mathcal{N}} \end{pmatrix} \quad (4)$$

is the Laplacian matrix of \mathcal{L} with

$$c_i \equiv \sum_{j=1}^{\mathcal{N}'} c_{ij}. \quad (5)$$

Here \vec{V} and \vec{I} are \mathcal{N} -vectors whose components are V_i and I_i respectively. We note in passing that if all nonzero resistances are equal to 1, then the off-diagonal elements of the matrix $-\mathbf{L}$ are precisely those of the adjacency matrix of \mathcal{L} .

² A study of random walks using matrices can be found in [3].

The Laplacian matrix \mathbf{L} of the network \mathcal{L} is also known as the Kirchhoff matrix, or simply the tree matrix; the latter name derived from the fact that all cofactors of \mathbf{L} are equal, and equal to the spanning tree generating function for \mathcal{L} [9].

To compute the resistance $R_{\alpha\beta}$ between two nodes α and β , we connect α and β to the two terminals of an external battery and measure the current I going through the battery while no other nodes are connected to external sources. Let the potentials at the two nodes be, respectively, V_α and V_β . Then, the desired resistance is the ratio

$$R_{\alpha\beta} = \frac{V_\alpha - V_\beta}{I}. \quad (6)$$

The computation of the two-point resistance $R_{\alpha\beta}$ is now reduced to solving (3) for V_α and V_β with the current given by

$$I_i = I(\delta_{i\alpha} - \delta_{i\beta}). \quad (7)$$

A probabilistic interpretation. The two-point resistance has a probabilistic interpretation. Consider a random walker walking on the network \mathcal{L} with the probability

$$p_{i \rightarrow j} = c_{ij}/c_i \quad (8)$$

of hopping from node i to node j , where $p_{i \rightarrow j}$ can be different from $p_{j \rightarrow i}$ (and thus the matrix \mathbf{L} is not necessarily symmetric). Let $P(\alpha; \beta)$ be the probability that the walker starting from node α will reach node β before returning to α , which is the probability of first passage. Then one has the relation [2, 5]

$$P(\alpha; \beta) = \frac{1}{c_\alpha R_{\alpha\beta}} \quad (9)$$

where c_α is defined in (5) and $R_{\alpha\beta}$ is obtained from (6) by solving (3) using the Laplacian defined by (8). If all resistances are 1, then (9) becomes

$$P(\alpha; \beta) = \frac{1}{z_\alpha R_{\alpha\beta}} \quad (10)$$

where z_α is the coordination number, or the number of nodes connected to, of the node α .

2. The two-point resistance: a theorem

Let Ψ_i and λ_i be the eigenvectors and eigenvalues of \mathbf{L} , namely,

$$\mathbf{L}\Psi_i = \lambda_i\Psi_i \quad i = 1, 2, \dots, \mathcal{N}.$$

Let $\psi_{i\alpha}$, $\alpha = 1, 2, \dots, \mathcal{N}$ be the components of Ψ_i . Since \mathbf{L} is Hermitian, the Ψ_i can be taken to be orthonormal satisfying

$$(\Psi_i^*, \Psi_j) = \sum_{\alpha} \psi_{i\alpha}^* \psi_{j\alpha} = \delta_{ij}.$$

Now the sum of all columns (or rows) of \mathbf{L} is identically zero, so one of the eigenvalues of \mathbf{L} is zero. It is readily verified that the zero eigenvalue $\lambda_1 = 0$ has the eigenvector

$$\psi_{1\alpha} = 1/\sqrt{\mathcal{N}} \quad \alpha = 1, 2, \dots, \mathcal{N}.$$

We now state our main result as a theorem.

Theorem. Consider a resistance network whose Laplacian has nonzero eigenvalues λ_i with orthonormal eigenvectors $\Psi_i = (\psi_{i1}, \psi_{i2}, \dots, \psi_{iN})$, $i = 2, 3, \dots, N$. Then the resistance between nodes α and β is given by

$$R_{\alpha\beta} = \sum_{i=2}^N \frac{1}{\lambda_i} |\psi_{i\alpha} - \psi_{i\beta}|^2. \quad (11)$$

Proof. We proceed to solve equation (3) by introducing the inverse of the Laplacian \mathbf{L} , or Green's function [6]. However, since one of the eigenvalues of \mathbf{L} is zero, the inverse of \mathbf{L} must be considered with care.

We circumvent this problem by adding a small term $\epsilon \mathbf{I}$ to the Laplacian, where \mathbf{I} is the $N \times N$ identity matrix, and set $\epsilon = 0$ at the end. The modified Laplacian

$$\mathbf{L}(\epsilon) = \mathbf{L} + \epsilon \mathbf{I}$$

is of the same form \mathbf{L} except that the diagonal elements c_i are replaced by $c_i + \epsilon$. It is clear that $\mathbf{L}(\epsilon)$ has eigenvalues $\lambda_i + \epsilon$ and is diagonalized by the same unitary transformation which diagonalizes \mathbf{L} .

The inverse of $\mathbf{L}(\epsilon)$, which is Green's function, is now well defined and we write

$$\mathbf{G}(\epsilon) = \mathbf{L}^{-1}(\epsilon).$$

Rewrite (3) as $\mathbf{L}(\epsilon) \vec{V}(\epsilon) = \vec{I}$ and multiply from the left by $\mathbf{G}(\epsilon)$ to obtain $\vec{V}(\epsilon) = \mathbf{G}(\epsilon) \vec{I}$. Explicitly, this reads

$$V_i(\epsilon) = \sum_{j=1}^N G_{ij}(\epsilon) I_j \quad i = 1, 2, \dots, N \quad (12)$$

where $G_{ij}(\epsilon)$ are the ij th elements of the matrix $\mathbf{G}(\epsilon)$.

We now compute Green's function $G_{ij}(\epsilon)$.

Let \mathbf{U} be the unitary matrix which diagonalizes $\mathbf{L}(\epsilon)$ and \mathbf{L} , namely,

$$\mathbf{U}^\dagger \mathbf{L} \mathbf{U} = \Lambda \quad \mathbf{U}^\dagger \mathbf{L}(\epsilon) \mathbf{U} = \Lambda(\epsilon). \quad (13)$$

It is readily verified that elements of \mathbf{U} are $U_{ij} = \psi_{ji}$, and Λ and $\Lambda(\epsilon)$ are, respectively, diagonal matrices with elements $\lambda_i \delta_{ij}$ and $(\lambda_i + \epsilon) \delta_{ij}$.

The inverse of the second equation in (13) is

$$\mathbf{U}^\dagger \mathbf{G}(\epsilon) \mathbf{U} = \Lambda^{-1}(\epsilon)$$

where $\Lambda^{-1}(\epsilon)$ has elements $(\lambda_i + \epsilon)^{-1} \delta_{ij}$. It follows that we have

$$\mathbf{G}(\epsilon) = \mathbf{U} \Lambda^{-1}(\epsilon) \mathbf{U}^\dagger$$

or, explicitly,

$$\begin{aligned} G_{\alpha\beta}(\epsilon) &= \sum_{i=1}^N U_{\alpha i} \left(\frac{1}{\lambda_i + \epsilon} \right) U_{\beta i}^* \\ &= \frac{1}{N\epsilon} + g_{\alpha\beta}(\epsilon) \end{aligned} \quad (14)$$

where

$$g_{\alpha\beta}(\epsilon) = \sum_{i=2}^N \frac{\psi_{i\alpha} \psi_{i\beta}^*}{\lambda_i + \epsilon}. \quad (15)$$

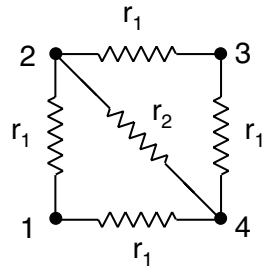


Figure 1. A network of four nodes.

The substitution of (14) into (12) now yields, after making use of the constraint (1),

$$V_i(\epsilon) = \sum_{j=1}^N g_{ij}(\epsilon) I_j.$$

It is now safe to take the $\epsilon \rightarrow 0$ limit in $g_{ij}(\epsilon)$ to obtain

$$V_i = \sum_{j=1}^N g_{ij}(0) I_j. \tag{16}$$

Finally, by combining (6), (7) with (16) we obtain

$$R_{\alpha\beta} = g_{\alpha\alpha}(0) + g_{\beta\beta}(0) - g_{\alpha\beta}(0) - g_{\beta\alpha}(0)$$

which becomes (11) after introducing (15). □

The usefulness of (11) is illustrated by the following examples.

Example 1. Consider the 4-node network shown in figure 1 with the Laplacian

$$\mathbf{L} = \begin{pmatrix} 2c_1 & -c_1 & 0 & -c_1 \\ -c_1 & 2c_1 + c_2 & -c_1 & -c_2 \\ 0 & -c_1 & 2c_1 & -c_1 \\ -c_1 & -c_2 & -c_1 & 2c_1 + c_2 \end{pmatrix}$$

where $c_1 = 1/r_1$, $c_2 = 1/r_2$. The nonzero eigenvalues of \mathbf{L} and their orthonormal eigenvectors are

$$\begin{aligned} \lambda_2 = 4c_1 & & \Psi_2 = \frac{1}{2}(1, -1, 1, -1) \\ \lambda_3 = 2c_1 & & \Psi_3 = \frac{1}{\sqrt{2}}(-1, 0, 1, 0) \\ \lambda_4 = 2(c_1 + c_2) & & \Psi_4 = \frac{1}{\sqrt{2}}(0, -1, 0, 1). \end{aligned}$$

Using (11), we obtain

$$\begin{aligned} R_{13} &= \frac{1}{\lambda_2}(\psi_{21} - \psi_{23})^2 + \frac{1}{\lambda_3}(\psi_{31} - \psi_{33})^2 + \frac{1}{\lambda_4}(\psi_{41} - \psi_{43})^2 = r_1 \\ R_{12} &= \frac{1}{\lambda_2}(\psi_{21} - \psi_{22})^2 + \frac{1}{\lambda_3}(\psi_{31} - \psi_{32})^2 + \frac{1}{\lambda_4}(\psi_{41} - \psi_{42})^2 = \frac{r_1(2r_1 + 3r_2)}{4(r_1 + r_2)}. \end{aligned}$$

Example 2. A complete graph is a network in which every node is connected to every other node. We consider $\mathcal{N}(\mathcal{N} - 1)/2$ resistors of equal resistance r embedded on a complete graph

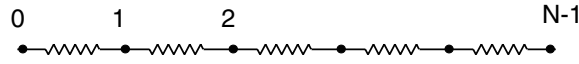


Figure 2. A one-dimensional network of \mathcal{N} nodes with free ends.

of \mathcal{N} nodes. The Laplacian is therefore

$$\mathbf{L}^{\text{complete graph}} = r^{-1} \begin{pmatrix} \mathcal{N}-1 & -1 & \cdots & -1 & -1 \\ -1 & \mathcal{N}-1 & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & \mathcal{N}-1 & -1 \\ -1 & -1 & \cdots & -1 & \mathcal{N}-1 \end{pmatrix}. \quad (17)$$

It is readily verified that the Laplacian (17) has eigenvalues $\lambda_0 = 0$ and $\lambda_n = \mathcal{N}r^{-1}$, $n = 1, 2, \dots, \mathcal{N}-1$, with corresponding eigenvectors Ψ_n having components

$$\psi_{n\alpha} = \frac{1}{\sqrt{\mathcal{N}}} \exp(i2\pi n\alpha/\mathcal{N}) \quad n, \alpha = 0, 1, \dots, \mathcal{N}-1.$$

It follows from (11) that the resistance between any two nodes α and β is

$$R_{\alpha,\beta} = r \sum_{n=1}^{\mathcal{N}} \frac{|\psi_{n\alpha} - \psi_{n\beta}|^2}{\mathcal{N}} = \frac{2}{\mathcal{N}}r. \quad (18)$$

In subsequent sections we consider applications of (11) to regular lattices.

3. One-dimensional lattice

It is instructive to first consider the one-dimensional case of a linear array of resistors, and to demonstrate that our formulation produces results implied by simple applications of Ohm's law. We consider free and periodic boundary conditions separately.

3.1. Free boundary condition

Consider $N-1$ resistors of resistance r each connected in series forming a chain of N nodes numbered $0, 1, 2, \dots, N-1$ as shown in figure 2, where each of the two end nodes connects to only one interior node. This is the Neumann (or the free) boundary condition.

The Laplacian (4) assumes the form

$$\mathbf{L}_{\{N \times 1\}}^{\text{free}} = r^{-1} \mathbf{T}_N^{\text{free}}$$

where $\mathbf{T}_N^{\text{free}}$ is the $N \times N$ matrix

$$\mathbf{T}_N^{\text{free}} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}. \quad (19)$$

The eigenvalues and eigenvectors of \mathbf{T}_N can be readily computed (see, for example, [10]), and are found to be

$$\lambda_n = 2(1 - \cos \phi_n) \quad n = 0, 1, \dots, N-1$$

$$\begin{aligned} \psi_{nx}^{(N)} &= \frac{1}{\sqrt{N}} & n = 0, \text{ for all } x \\ &= \sqrt{\frac{2}{N}} \cos((x + 1/2)\phi_n) & n = 1, 2, \dots, N - 1, \text{ for all } x \end{aligned} \tag{20}$$

where $\phi_n = n\pi/N$. Thus, using (11), the resistance between nodes x_1 and x_2 is

$$\begin{aligned} R_{\{N \times 1\}}^{\text{free}}(x_1, x_2) &= \frac{r}{N} \sum_{n=1}^{N-1} \frac{[\cos(x_1 + \frac{1}{2})\phi_n - \cos(x_2 + \frac{1}{2})\phi_n]^2}{1 - \cos \phi_n} \\ &= r \left[F_N(x_1 + x_2 + 1) + F_N(x_1 - x_2) - \frac{1}{2}F_N(2x_1 + 1) - \frac{1}{2}F_N(2x_2 + 1) \right] \end{aligned} \tag{21}$$

where

$$F_N(\ell) = \frac{1}{N} \sum_{n=1}^{N-1} \frac{1 - \cos(\ell\phi_n)}{1 - \cos \phi_n}. \tag{22}$$

Note that without loss of generality we can take $0 \leq \ell < 2N$.

The function $F_N(\ell)$ can be evaluated by taking the limit $\lambda \rightarrow 0$ of the function $I_1(0) - I_1(\ell)$ evaluated in (61) below. It is, however, instructive to evaluate $F_N(\ell)$ directly. To do this we consider the real part of the summation

$$T_N(\ell) \equiv \frac{1}{N} \sum_{n=1}^{N-1} \frac{1 - \exp(i\ell\phi_n)}{1 - \exp(i\phi_n)}.$$

First, writing out the real part of the summand, we obtain

$$\begin{aligned} \text{Re } T_N(\ell) &= \frac{1}{N} \sum_{n=1}^{N-1} \frac{1 - \cos \phi_n - \cos \ell\phi_n + \cos(\ell - 1)\phi_n}{2(1 - \cos \phi_n)} \\ &= \frac{1}{2} [F_N(1) + F_N(\ell) - F_N(\ell - 1)]. \end{aligned} \tag{23}$$

Second, we expand the summand to obtain

$$T_N(\ell) = \frac{1}{N} \sum_{n=1}^{N-1} \sum_{\ell'=0}^{\ell-1} \exp(i\pi n\ell'/N)$$

and carry out the summation over n . The term $\ell' = 0$ yields $F_N(1)$ and terms $\ell' \geq 1$ can be explicitly summed, leading to

$$T_N(\ell) = F_N(1) + \frac{1}{N} \sum_{\ell'=1}^{\ell-1} \left[\frac{1 - (-1)^{\ell'}}{1 - \exp(i\pi\ell'/N)} - 1 \right] \quad \ell < 2N. \tag{24}$$

We now evaluate the real part of $T_N(\ell)$ given by (24). Using the identity

$$\text{Re} \left(\frac{1}{1 - e^{i\theta}} \right) = \frac{1}{2} \quad 0 < \theta < 2\pi \tag{25}$$

we find

$$\text{Re } T_N(\ell) = F_N(1) - \frac{1}{4N} [2\ell - 3 - (-1)^\ell]. \tag{26}$$

Equating (23) with (26) and noting $F_N(1) = 1 - 1/N$, we are led to the recursion relation

$$F_N(\ell) - F_N(\ell - 1) = 1 - \frac{1}{2N} [2\ell - 1 - (-1)^\ell]$$

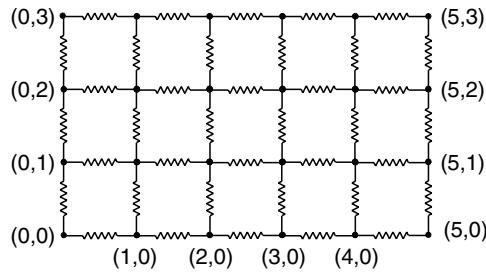


Figure 4. A 5×4 rectangular network.

Again, the function $G_N(\ell)$ can be evaluated by taking the limit $\lambda \rightarrow 0$ of $I_2(0) - I_2(\ell)$ given in (62) below. Alternately, it can be evaluated directly as in section 3.1 above by considering the real part of the summation

$$\frac{1}{N} \sum_{n=1}^{N-1} \frac{1 - \exp(i2\ell\phi_n)}{1 - \exp(i2\phi_n)}$$

in two different ways. This leads to the recursion relation

$$G_N(\ell) - G_N(\ell - 1) = 1 - \frac{1}{N}(2\ell - 1)$$

which can be solved to yield

$$G_N(\ell) = |\ell| - \ell^2/N. \tag{32}$$

It follows that we have

$$R_{\{N \times 1\}}^{\text{per}}(x_1, x_2) = r|x_1 - x_2| \left[1 - \frac{|x_1 - x_2|}{N} \right]. \tag{33}$$

Expression (33) is the expected resistance of two resistors $|x_1 - x_2|r$ and $(N - |x_1 - x_2|)r$ connected in parallel as in a ring.

4. Two-dimensional network: free boundaries

Consider a rectangular network of resistors connected in an array of $M \times N$ nodes forming a network with free boundaries as shown in figure 4.

Number the nodes by coordinates $\{m, n\}$, $0 \leq m \leq M - 1$, $0 \leq n \leq N - 1$ and denote the resistances along the two principal directions by r and s . The Laplacian is therefore

$$\mathbf{L}_{\{M \times N\}}^{\text{free}} = r^{-1} \mathbf{T}_M^{\text{free}} \otimes \mathbf{I}_N + s^{-1} \mathbf{I}_M \otimes \mathbf{T}_N^{\text{free}} \tag{34}$$

where \otimes denotes direct matrix products and $\mathbf{T}_N^{\text{free}}$ is given by (19). The Laplacian can be diagonalized in the two subspaces separately, yielding eigenvalues

$$\lambda_{(m,n)} = 2r^{-1}(1 - \cos \theta_m) + 2s^{-1}(1 - \cos \phi_n) \tag{35}$$

and eigenvectors

$$\psi_{(m,n);(x,y)}^{\text{free}} = \psi_{m_x}^{(M)} \psi_{n_y}^{(N)}. \tag{36}$$

It then follows from (11) that the resistance R_{free} between two nodes $\mathbf{r}_1 = (x_1, y_1)$ and $\mathbf{r}_2 = (x_2, y_2)$ is

$$\begin{aligned} R_{\{M \times N\}}^{\text{free}}(\mathbf{r}_1, \mathbf{r}_2) &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \frac{|\psi_{(m,n);(x_1,y_1)}^{\text{free}} - \psi_{(m,n);(x_2,y_2)}^{\text{free}}|^2}{\lambda_{(m,n)}} \\ &= \frac{r}{N}|x_1 - x_2| + \frac{s}{M}|y_1 - y_2| + \frac{2}{MN} \\ &\quad \times \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} \frac{[\cos(x_1 + \frac{1}{2})\theta_m \cos(y_1 + \frac{1}{2})\phi_n - \cos(x_2 + \frac{1}{2})\theta_m \cos(y_2 + \frac{1}{2})\phi_n]^2}{r^{-1}(1 - \cos \theta_m) + s^{-1}(1 - \cos \phi_n)} \end{aligned} \quad (37)$$

where

$$\theta_m = \frac{m\pi}{M} \quad \phi_n = \frac{n\pi}{N}.$$

Here, use has been made of (28) for summing over the $m = 0$ and $n = 0$ terms. As expected, the resulting expression (37) now depends on the coordinates x_1, y_1, x_2, y_2 of the two nodes explicitly.

The usefulness of (37) is best illustrated by applications. Several examples are now given.

Example 3. For $M = 5, N = 4, r = s$, the resistance between nodes $\{0, 0\}$ and $\{3, 3\}$ is computed from (37) as

$$\begin{aligned} R_{\{5 \times 4\}}^{\text{free}}(\{0, 0\}, \{3, 3\}) &= \left(\frac{3}{4} + \frac{3}{5} + \frac{9877\,231}{27\,600\,540} \right) r \\ &= (1.70786\dots)r. \end{aligned} \quad (38)$$

Example 4. For $M = N = 4$, we find

$$R_{\{4 \times 4\}}^{\text{free}}(\{0, 0\}, \{3, 3\}) = \frac{(r+s)(r^2+5rs+s^2)(3r^2+7rs+3s^2)}{2(2r^2+4rs+s^2)(r^2+4rs+2s^2)}. \quad (39)$$

Example 5. We evaluate the resistance between two nodes in the interior of a large lattice. Consider, for definiteness, both $M, N = \text{odd}$ (for other parities the result (40) below is the same) and compute the resistance between two nodes in the centre region,

$$\begin{aligned} \mathbf{r}_1 &= (x_1, y_1) = \left(\frac{M-1}{2} + p_1, \frac{N-1}{2} + q_1 \right) \\ \mathbf{r}_2 &= (x_2, y_2) = \left(\frac{M-1}{2} + p_2, \frac{N-1}{2} + q_2 \right) \end{aligned}$$

where $p_i, q_i \ll M, N$ are integers. The numerator of the summand in (37) becomes

$$\begin{aligned} &\left[\cos\left(\frac{m\pi}{2} + p_1\theta_m\right) \cos\left(\frac{n\pi}{2} + q_1\phi_n\right) - \cos\left(\frac{m\pi}{2} + p_2\theta_m\right) \cos\left(\frac{n\pi}{2} + q_2\phi_n\right) \right]^2 \\ &= (\cos p_1\theta_m \cos q_1\phi_n - \cos p_2\theta_m \cos q_2\phi_n)^2 && m, n = \text{even} \\ &= (\sin p_1\theta_m \sin q_1\phi_n - \sin p_2\theta_m \sin q_2\phi_n)^2 && m, n = \text{odd} \\ &= (\sin p_1\theta_m \cos q_1\phi_n - \sin p_2\theta_m \cos q_2\phi_n)^2 && m = \text{odd}, n = \text{even} \\ &= (\cos p_1\theta_m \sin q_1\phi_n - \cos p_2\theta_m \sin q_2\phi_n)^2 && m = \text{even}, n = \text{odd} \end{aligned}$$

and the summation in (37) breaks into four parts. In the $M, N \rightarrow \infty$ limit the summations can be replaced by integrals. After some reduction we arrive at the expression

$$R_{\infty}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{(2\pi)^2} \int_0^{2\pi} d\phi \int_0^{2\pi} d\theta \left(\frac{1 - \cos(x_1 - x_2)\theta \cos(y_1 - y_2)\phi}{r^{-1}(1 - \cos \theta) + s^{-1}(1 - \cos \phi)} \right). \quad (40)$$

Our expression (40) agrees with the known expression obtained previously [1]. It can be verified that expression (40) holds between any two nodes in the lattice, provided that the two nodes are far from the boundaries.

5. Two-dimensional network: periodic boundary conditions

We next consider an $M \times N$ network with periodic boundary conditions. The Laplacian in this case is

$$\mathbf{L}_{\{M \times N\}}^{\text{per}} = r^{-1} \mathbf{T}_M^{\text{per}} \otimes \mathbf{I}_N + s^{-1} \mathbf{I}_M \otimes \mathbf{T}_N^{\text{per}} \tag{41}$$

where $\mathbf{T}_N^{\text{per}}$ is given by (29). The Laplacian (41) can again be diagonalized in the two subspaces separately, yielding eigenvalues and eigenvectors

$$\begin{aligned} \lambda_{(m,n)} &= 2r^{-1}(1 - \cos 2\theta_m) + 2s^{-1}(1 - \cos 2\phi_n) \\ \psi_{(m,n);(x,y)} &= \frac{1}{\sqrt{MN}} \exp(i2x\theta_m) \exp(i2y\phi_n). \end{aligned} \tag{42}$$

This leads to the resistance between nodes $\mathbf{r}_1 = (x_1, y_1)$ and $\mathbf{r}_2 = (x_2, y_2)$,

$$\begin{aligned} R_{\{M \times N\}}^{\text{per}}(\mathbf{r}_1, \mathbf{r}_2) &= \sum_{m=0}^{M-1} \sum_{\substack{n=0 \\ (m,n) \neq (0,0)}}^{N-1} \frac{|\psi_{(m,n);(x_1,y_1)} - \psi_{(m,n);(x_2,y_2)}|^2}{\lambda_{(m,n)}} \\ &= \frac{r}{N} \left[|x_1 - x_2| - \frac{(x_1 - x_2)^2}{M} \right] + \frac{s}{M} \left[|y_1 - y_2| - \frac{(y_1 - y_2)^2}{N} \right] \\ &\quad + \frac{1}{MN} \sum_{m=1}^M \sum_{n=1}^N \frac{1 - \cos[2(x_1 - x_2)\theta_m + 2(y_1 - y_2)\phi_n]}{r^{-1}(1 - \cos 2\theta_m) + s^{-1}(1 - \cos 2\phi_n)} \end{aligned} \tag{43}$$

where the two terms in the second line are given by (33). It is clear that the result depends only on the differences $|x_1 - x_2|$ and $|y_1 - y_2|$, as it should under periodic boundary conditions.

Example 6. Using (43) the resistance between nodes $\{0, 0\}$ and $\{3, 3\}$ on a 5×4 periodic lattice with $r = s$ is

$$\begin{aligned} R_{\{5 \times 4\}}^{\text{per}}(\{0, 0\}, \{3, 3\}) &= \left(\frac{3}{10} + \frac{3}{20} + \frac{1799}{7790} \right) r \\ &= (0.680937\dots)r. \end{aligned} \tag{44}$$

This is to be compared to the value $(1.707863\dots)r$ for free boundary conditions given in example 3. It can also be verified that the resistance between nodes $\{0, 0\}$ and $\{2, 1\}$ is also given by (44) as it must for a periodic lattice.

In the limit of $M, N \rightarrow \infty$ with $|\mathbf{r}_1 - \mathbf{r}_2|$ finite, (43) becomes

$$\begin{aligned} R_{\infty}(\mathbf{r}_1, \mathbf{r}_2) &= \frac{1}{(2\pi)^2} \int_0^{2\pi} d\phi \int_0^{2\pi} d\theta \frac{1 - \cos[(x_1 - x_2)\theta + (y_1 - y_2)\phi]}{r^{-1}(1 - \cos \theta) + s^{-1}(1 - \cos \phi)} \\ &= \frac{1}{(2\pi)^2} \int_0^{2\pi} d\phi \int_0^{2\pi} d\theta \frac{1 - \cos(x_1 - x_2)\theta \cos(y_1 - y_2)\phi}{r^{-1}(1 - \cos \theta) + s^{-1}(1 - \cos \phi)} \end{aligned} \tag{45}$$

which agrees with (40).

6. Cylindrical boundary conditions

Consider an $M \times N$ resistor network embedded on a cylinder with periodic boundary in the direction of M and free boundaries in the direction of N .

The Laplacian is

$$\mathbf{L}_{\{M \times N\}}^{\text{cyl}} = r^{-1} \mathbf{T}_M^{\text{per}} \otimes \mathbf{I}_N + s^{-1} \mathbf{I}_M \otimes \mathbf{T}_N^{\text{free}}$$

which can again be diagonalized in the two subspaces separately. This gives the eigenvalues and eigenvectors

$$\begin{aligned} \lambda_{(m,n)} &= 2r^{-1}(1 - \cos 2\theta_m) + 2s^{-1}(1 - \cos \phi_n) \\ \psi_{(m,n);(x,y)}^{\text{cyl}} &= \frac{1}{\sqrt{M}} \exp(i2x\theta_m) \psi_{ny}^{(N)}. \end{aligned}$$

It follows that the resistance R_{free} between nodes $\mathbf{r}_1 = (x_1, y_1)$ and $\mathbf{r}_2 = (x_2, y_2)$ is

$$\begin{aligned} R_{\{M \times N\}}^{\text{cyl}}(\mathbf{r}_1, \mathbf{r}_2) &= \sum_{m=0}^{M-1} \sum_{\substack{n=0 \\ (m,n) \neq (0,0)}}^{N-1} \frac{|\psi_{(m,n);(x_1,y_1)}^{\text{cyl}} - \psi_{(m,n);(x_2,y_2)}^{\text{cyl}}|^2}{\lambda_{(m,n)}} \\ &= \frac{r}{N} \left[|x_1 - x_2| - \frac{(x_1 - x_2)^2}{M} \right] + \frac{s}{M} |y_1 - y_2| \\ &\quad + \frac{2}{MN} \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} \left(\frac{C_1^2 + C_2^2 - 2C_1 C_2 \cos 2(x_1 - x_2)\theta_m}{r^{-1}(1 - \cos 2\theta_m) + s^{-1}(1 - \cos \phi_n)} \right) \end{aligned}$$

where

$$C_1 = \cos\left(y_1 + \frac{1}{2}\right)\phi_n \quad C_2 = \cos\left(y_2 + \frac{1}{2}\right)\phi_n. \quad (46)$$

It can be verified that in the $M, N \rightarrow \infty$ limit (46) leads to the same expression (40) for two interior nodes in an infinite lattice.

Example 7. The resistance between nodes $\{0, 0\}$ and $\{3, 3\}$ on a 5×4 cylindrical lattice with $r = s$ is computed to be

$$\begin{aligned} R_{\{5 \times 4\}}^{\text{cyl}}(\{0, 0\}, \{3, 3\}) &= \left(\frac{3}{10} + \frac{3}{5} + \frac{5023}{8835} \right) r \\ &= (1.46853\dots)r. \end{aligned} \quad (47)$$

This is compared to the values of $(1.70786\dots)r$ for free boundary conditions and $(0.680937\dots)r$ for periodic boundary conditions.

7. Möbius strip

We next consider an $M \times N$ resistor lattice embedded on a Möbius strip of width N and length M , which is a rectangular strip connected at two ends after a 180° twist of one of the two ends of the strip. The schematic figure of a Möbius strip is shown in figure 5(a). The Laplacian for this lattice assumes the form

$$\mathbf{L}_{\{M \times N\}}^{\text{Mob}} = r^{-1} [\mathbf{H}_M \otimes \mathbf{I}_N - \mathbf{K}_M \otimes \mathbf{J}_N] + s^{-1} \mathbf{I}_M \otimes \mathbf{T}_N^{\text{free}} \quad (48)$$

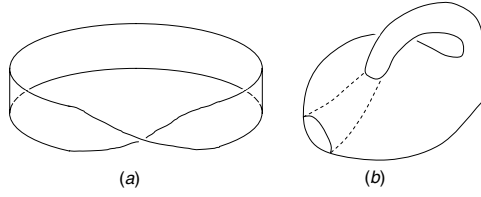


Figure 5. (a) The schematic plot of an Möbius strip. (b) The schematic plot of a Klein bottle.

where

$$\mathbf{K}_N = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix} \quad \mathbf{J}_N = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

$$\mathbf{H}_N = \mathbf{T}_N^{\text{per}} + \mathbf{K}_N = \begin{pmatrix} 2 & -1 & \dots & 0 & 0 \\ -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & \dots & -1 & 2 \end{pmatrix}$$

are $N \times N$ matrices. Now $\mathbf{T}_N^{\text{free}}$ and \mathbf{J}_N commute so they can be replaced by their respective eigenvalues $2(1 - \cos \phi_n)$ and $(-1)^n$ and we need only diagonalize an $M \times M$ matrix. This leads to the following eigenvalues and eigenvectors of the Laplacian (48) [10]³:

$$\lambda_{(m,n)} = 2r^{-1} \cos \left[(4m + 1 - (-1)^n) \frac{\pi}{2M} \right] + 2s^{-1} \left(1 - \cos \frac{n\pi}{N} \right) \tag{49}$$

$$\psi_{(m,n);(x,y)}^{\text{Mob}} = \frac{1}{\sqrt{M}} \exp \left[i(4m + 1 - (-1)^n) \frac{x\pi}{2M} \right] \psi_{ny}^{(N)}$$

where $\psi_{ny}^{(N)}$ is given in (20). Substituting these expressions into (11) and after a little reduction, we obtain

$$R_{\{M \times N\}}^{\text{Mob}}(\mathbf{r}_1, \mathbf{r}_2) = \sum_{m=0}^{M-1} \sum_{n=0, (m,n) \neq (0,0)}^{N-1} \frac{|\psi_{(m,n);(x_1,y_1)}^{\text{Mob}} - \psi_{(m,n);(x_2,y_2)}^{\text{Mob}}|^2}{\lambda_{(m,n)}}$$

$$= \frac{r}{N} \left[|x_1 - x_2| - \frac{(x_1 - x_2)^2}{M} \right]$$

$$+ \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=1}^{N-1} \frac{C_1^2 + C_2^2 - 2C_1 C_2 \cos \left[(x_1 - x_2) (4m + 1 - (-1)^n) \frac{\pi}{2M} \right]}{r^{-1} \left[1 - \cos(4m + 1 - (-1)^n) \frac{\pi}{2M} \right] + s^{-1} (1 - \cos \phi_n)} \tag{50}$$

where C_1 and C_2 have been given in (46).

Example 8. The 2×2 Möbius strip is a complete graph of $\mathcal{N} = 4$ nodes. For $r = s$ expression (50) gives a resistance $r/2$ between any two nodes which agrees with (18).

³ I am indebted to W-J Tzeng for working out (49) and (53).

Example 9. The resistance between nodes $(0, 0)$ and $(3, 3)$ on a 5×4 Möbius strip with $r = s$ is computed from (50) as

$$\begin{aligned} R_{\{5 \times 4\}}^{\text{Mob}}(\{0, 0\}, \{3, 3\}) &= \left(\frac{3}{10} + \frac{1609}{2698} \right) r \\ &= (0.896367\dots)r. \end{aligned} \quad (51)$$

This is to be compared to the corresponding values for the same network under other boundary conditions in examples 3, 6 and 7.

8. Klein bottle

A Klein bottle is a Möbius strip with a periodic boundary condition imposed in the other direction, a schematic diagram of which is shown in figure 5(b). We consider an $M \times N$ resistor grid embedded on a Klein bottle.

Let the network have a twisted boundary condition in the direction of the length M and a periodic boundary condition in the direction of the width N . Then, analogous to (48), the Laplacian of the network assumes the form

$$\mathbf{L}_{\{M \times N\}}^{\text{Klein}} = r^{-1}[\mathbf{H}_M \otimes \mathbf{I}_N - \mathbf{K}_M \otimes \mathbf{J}_N] + s^{-1}\mathbf{I}_M \otimes \mathbf{T}_N^{\text{per}}. \quad (52)$$

Now the matrices \mathbf{J}_N and $\mathbf{T}_N^{\text{per}}$ commute so they can be replaced by their respective eigenvalues ± 1 and $2(1 - \cos 2\phi_n)$ in (52) and one need only diagonalize an $M \times M$ matrix. This leads to the following eigenvalues and eigenvectors for $\mathbf{L}_{\{M \times N\}}^{\text{Klein}}$ [10] (see footnote 3):

$$\begin{aligned} \lambda_{(m,n)}(\tau) &= 2r^{-1} \left[1 - \cos \left((2m + \tau) \frac{\pi}{M} \right) \right] + 2s^{-1} \left(1 - \cos \frac{2n\pi}{N} \right) \\ \psi_{(m,n);(x,y)}^{\text{Klein}} &= \frac{1}{\sqrt{M}} \exp \left[i(2m + \tau) \frac{x\pi}{M} \right] \psi_{ny}^{(N)\dagger} \end{aligned} \quad (53)$$

where

$$\begin{aligned} \tau = \tau_n &= 0 & n = 0, 1, \dots, \left[\frac{N-1}{2} \right] \\ &= 1 & n = \left[\frac{N+1}{2} \right], \dots, N-1 \end{aligned}$$

and

$$\begin{aligned} \psi_{ny}^{(N)\dagger} &= \frac{1}{\sqrt{N}} & n = 0 \\ &= \sqrt{\frac{2}{N}} \cos \left[(2y+1) \frac{n\pi}{N} \right] & n = 1, 2, \dots, \left[\frac{N-1}{2} \right] \\ &= \frac{1}{\sqrt{N}} (-1)^y & n = \frac{N}{2} & \text{for even } N \text{ only} \\ &= \sqrt{\frac{2}{N}} \sin \left[(2y+1) \frac{n\pi}{N} \right] & n = \left[\frac{N}{2} \right] + 1, \dots, N-1. \end{aligned}$$

Substituting these expressions into (11), separating out the summation for $n = 0$, and making use of the identity

$$\sin \left[(2y+1) \left(\frac{N}{2} + n \right) \frac{\pi}{N} \right] = (-1)^y \cos \left[(2y+1) \frac{n\pi}{N} \right]$$

we obtain after some reduction

$$\begin{aligned}
 R_{\{M \times N\}}^{\text{Klein}}(\mathbf{r}_1, \mathbf{r}_2) &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \frac{|\psi_{(m,n);(x_1,y_1)}^{\text{Klein}} - \psi_{(m,n);(x_2,y_2)}^{\text{Klein}}|^2}{\lambda_{(m,n)}(\tau_n)} \\
 &= \frac{r}{N} \left[|x_1 - x_2| - \frac{(x_1 - x_2)^2}{M} \right] \\
 &\quad + \sum_{m=0}^{M-1} \sum_{n=1}^{N-1} \frac{1}{\lambda_{(m,n)}(\tau_n)} |\psi_{(m,n);(x_1,y_1)}^{\text{Klein}} - \psi_{(m,n);(x_2,y_2)}^{\text{Klein}}|^2 \\
 &= \frac{r}{N} \left[|x_1 - x_2| - \frac{(x_1 - x_2)^2}{M} \right] + \Delta_N + \frac{2}{MN} \sum_{\tau=0}^1 \sum_{m=0}^{M-1} \sum_{n=1}^{\lfloor \frac{N-1}{2} \rfloor} \\
 &\quad \times \frac{C_1^2 + C_2^2 - 2(-1)^{(y_1-y_2)\tau} C_1 C_2 \cos[2(x_1 - x_2)\Theta_m(\tau)]}{\lambda_{(m,n)}(\tau)} \tag{54}
 \end{aligned}$$

where

$$\begin{aligned}
 \Theta_m(\tau) &= \left(m + \frac{\tau}{2}\right) \frac{\pi}{M} \\
 \Delta_N &= \frac{2}{MN} \sum_{m=0}^{M-1} \frac{1 - (-1)^{y_1-y_2} \cos[2(x_1 - x_2)\Theta_m(1)]}{\lambda_{(m,N/2)}(1)} \quad N = \text{even} \tag{55} \\
 &= 0 \quad N = \text{odd}
 \end{aligned}$$

and $C_i = \cos[(y_i + 1/2)n\pi/N]$, $i = 1, 2$, as defined in (46).

Example 10. The resistance between nodes $(0, 0)$ and $(3, 3)$ on a 5×4 ($N = \text{even}$) Klein bottle with $r = s$ is computed from (54) as

$$\begin{aligned}
 R_{\{5 \times 4\}}^{\text{Klein}}(\{0, 0\}, \{3, 3\}) &= \left(\frac{3}{10} + \frac{5}{58} + \frac{56}{209}\right)r \\
 &= (0.654149 \dots)r \tag{56}
 \end{aligned}$$

where the three terms in the first line are from the evaluation of corresponding terms in (54). The result is to be compared to the corresponding value for the same 5×4 network under the Möbius boundary condition considered in example 9, which is the Klein bottle without periodic boundary connections.

9. Higher dimensional lattices

The two-point resistance can be computed using (11) for lattices in any spatial dimensionality under various boundary conditions. To illustrate, we give the result for an $M \times N \times L$ cubic lattice with free boundary conditions.

Number the nodes by $\{m, n, \ell\}$, $0 \leq m \leq M - 1$, $0 \leq n \leq N - 1$, $0 \leq \ell \leq L - 1$, and let the resistances along the principal axes be, respectively, r , s and t . The Laplacian then assumes the form

$$\mathbf{L}_{\{M \times N \times L\}}^{\text{free}} = r^{-1} \mathbf{T}_M^{\text{free}} \otimes \mathbf{I}_N \otimes \mathbf{I}_L + s^{-1} \mathbf{I}_M \otimes \mathbf{T}_N^{\text{free}} \otimes \mathbf{I}_L + t^{-1} \mathbf{I}_M \otimes \mathbf{I}_N \otimes \mathbf{T}_L^{\text{free}}$$

where $\mathbf{T}_N^{\text{free}}$ is given by (19). The Laplacian can be diagonalized in the three subspaces separately, yielding eigenvalues

$$\lambda_{(m,n,\ell)} = 2r^{-1}(1 - \cos \theta_m) + 2s^{-1}(1 - \cos \phi_n) + 2t^{-1}(1 - \cos \alpha_\ell) \tag{57}$$

and eigenvectors

$$\psi_{(m,n,\ell);(x,y,z)}^{\text{free}} = \psi_{mx}^{(M)} \psi_{ny}^{(N)} \psi_{\ell z}^{(L)}$$

where $\psi_{mx}^{(M)}$ is given by (20) and $\alpha_\ell = \ell\pi/L$. It then follows from (11) that the resistance R_{free} between two nodes $\mathbf{r}_1 = (x_1, y_1, z_1)$ and $\mathbf{r}_2 = (x_2, y_2, z_2)$ is

$$R_{\{M \times N \times L\}}^{\text{free}}(\mathbf{r}_1, \mathbf{r}_2) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \sum_{\ell=0}^{L-1} \lambda_{(m,n,\ell)}^{-1} \left| \psi_{(m,n,\ell);(x_1,y_1,z_1)}^{\text{free}} - \psi_{(m,n,\ell);(x_2,y_2,z_2)}^{\text{free}} \right|^2. \quad (58)$$

The summation can be broken down as

$$\begin{aligned} R_{\{M \times N \times L\}}^{\text{free}}(\mathbf{r}_1, \mathbf{r}_2) &= \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} \sum_{\ell=1}^{L-1} \frac{\left| \psi_{(m,n,\ell);(x_1,y_1,z_1)}^{\text{free}} - \psi_{(m,n,\ell);(x_2,y_2,z_2)}^{\text{free}} \right|^2}{\lambda_{(m,n,\ell)}} \\ &+ \frac{1}{L} R_{\{M \times N\}}^{\text{free}}(\{x_1, y_1\}, \{x_2, y_2\}) + \frac{1}{M} R_{\{N \times L\}}^{\text{free}}(\{y_1, z_1\}, \{y_2, z_2\}) \\ &+ \frac{1}{N} R_{\{L \times M\}}^{\text{free}}(\{z_1, x_1\}, \{z_2, x_2\}) - \frac{1}{MN} R_{\{L \times 1\}}^{\text{free}}(x_1, x_2) \\ &- \frac{1}{NL} R_{\{M \times 1\}}^{\text{free}}(y_1, y_2) - \frac{1}{LM} R_{\{N \times 1\}}^{\text{free}}(z_1, z_2). \end{aligned} \quad (59)$$

All terms in (59) have previously been computed except the summation in the first line.

Example 11. The resistance between the nodes (0, 0, 0) and (3, 3, 3) in a $5 \times 5 \times 4$ lattice with free boundaries and $r = s = t$ is computed from (59) as

$$\begin{aligned} R_{\{5 \times 5 \times 4\}}^{\text{free}}(\{0, 0, 0\}; \{3, 3, 3\}) &= \left(\frac{327\,687\,658\,482\,872}{352\,468\,567\,489\,225} \right) r \\ &= (0.929\,693\dots)r. \end{aligned} \quad (60)$$

Example 12. The resistance between two interior nodes \mathbf{r}_1 and \mathbf{r}_2 can be worked out as in example 5. The result is

$$\begin{aligned} R_\infty(\mathbf{r}_1, \mathbf{r}_2) &= \frac{1}{(2\pi)^3} \int_0^{2\pi} d\phi \int_0^{2\pi} d\theta \int_0^{2\pi} d\alpha \\ &\times \left(\frac{1 - \cos(x_1 - x_2)\theta \cos(y_1 - y_2)\phi \cos(z_1 - z_2)\alpha}{r^{-1}(1 - \cos\theta) + s^{-1}(1 - \cos\phi) + t^{-1}(1 - \cos\alpha)} \right) \end{aligned}$$

which is the known result [1].

10. Summation and product identities

The reduction of the two-point resistances for one-dimensional lattices to the simple and familiar expressions of (28) and (33) is facilitated by the use of the summation identities (27) and (32). In this section, we extend the consideration and deduce generalizations of these identities which can be used to reduce the computational labour for lattice sums as well as analyse large-size expansions in two and higher dimensions.

We state two new lattice sum identities as a proposition.

Proposition. Define

$$I_\alpha(\ell) = \frac{1}{N} \sum_{n=0}^{N-1} \frac{\cos(\alpha \ell \frac{n\pi}{N})}{\cosh \lambda - \cos(\alpha \frac{n\pi}{N})} \quad \alpha = 1, 2.$$

Then the following identities hold for λ real and $N = 1, 2, \dots$:

$$I_1(\ell) = \frac{\cosh(N - \ell)\lambda}{(\sinh \lambda) \sinh(N\lambda)} + \frac{1}{N} \left[\frac{1}{\sinh^2 \lambda} + \frac{1 - (-1)^\ell}{4 \cosh^2(\lambda/2)} \right] \quad 0 \leq \ell < 2N \tag{61}$$

$$I_2(\ell) = \frac{\cosh\left(\frac{N}{2} - \ell\right)\lambda}{(\sinh \lambda) \sinh(N\lambda/2)} \quad 0 \leq \ell < N. \tag{62}$$

Remarks.

- (1) It is clear that without the loss of generality we can restrict ℓ to the ranges indicated.
- (2) For $\ell = 0$ and $\lambda \rightarrow 0$, $I_1(0)$ leads to (27) and $I_2(0)$ leads to (32).
- (3) In the $N \rightarrow \infty$ limit both (61) and (62) become the integral⁴

$$\frac{1}{\pi} \int_0^\pi \frac{\cos(\ell\theta)}{\cosh \lambda - \cos \theta} d\theta = \frac{e^{-\ell|\lambda|}}{\sinh |\lambda|} \quad \ell \geq 0. \tag{63}$$

- (4) Set $\ell = 0$ in (61), multiplying by $\sinh \lambda$ and integrating over λ , we obtain the product identity [11]

$$\prod_{n=0}^{N-1} \left(\cosh \lambda - \cos \frac{n\pi}{N} \right) = (\sinh N\lambda) \tanh(\lambda/2). \tag{64}$$

- (5) Set $\ell = 0$ in (62), multiplying by $\sinh \lambda$ and integrating over λ , we obtain the product identity

$$\prod_{n=0}^{N-1} \left(\cosh \lambda - \cos \frac{2n\pi}{N} \right) = \sinh^2(N\lambda/2). \tag{65}$$

Proof of the proposition. It is convenient to introduce the notation

$$S_\alpha(\ell) = \frac{1}{N} \sum_{n=0}^{N-1} \frac{\cos(\ell\theta_n)}{1 + a^2 - 2a \cos \theta_n} \quad a < 1 \quad \alpha = 1, 2 \tag{66}$$

so that

$$I_\alpha(\ell) = 2a S_\alpha(\ell) \quad a = e^{-|\lambda|}. \tag{67}$$

It is readily seen that we have the identity

$$S_\alpha(1) = \frac{1}{2a} [(1 + a^2) S_\alpha(0) - 1]. \tag{68}$$

Proof of (61). First we evaluate $S_1(0)$ by carrying out the following summation, where Re denotes the real part, in two different ways. First we have

$$\begin{aligned} \text{Re} \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{1 - a e^{i\theta_n}} &= \text{Re} \frac{1}{N} \sum_{n=0}^{N-1} \frac{1 - a e^{-i\theta_n}}{|1 - a e^{i\theta_n}|^2} \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \frac{1 - a \cos \theta_n}{1 + a^2 - 2a \cos \theta_n} \\ &= S_1(0) - a S_1(1) \\ &= \frac{1}{2} [1 + (1 - a^2) S_1(0)]. \end{aligned} \tag{69}$$

⁴ This integral is equivalent to the integral (A6) of [1], where it is evaluated using the method of residues.

Secondly, by expanding the summand we have

$$\operatorname{Re} \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{1 - a e^{i\theta_n}} = \operatorname{Re} \frac{1}{N} \sum_{n=0}^{N-1} \sum_{\ell=0}^{\infty} a^\ell \exp(i\ell n\pi/N)$$

and carry out the summation over n for fixed ℓ . It is clear that all $\ell = \text{even}$ terms vanish except those with $\ell = 2mN$, $m = 0, 1, 2, \dots$ which yield $\sum_{m=0}^{\infty} a^{2mN} = 1/(1 - a^{2N})$. For $\ell = \text{odd} = 2m + 1$, $m = 0, 1, 2, \dots$ we have

$$\operatorname{Re} \sum_{n=0}^{N-1} \exp(i(2m+1)n\pi/N) = \operatorname{Re} \frac{1 - (-1)^{2m+1}}{1 - \exp(i(2m+1)\pi/N)} = 1$$

after making use of (25). So the summation over $\ell = \text{odd}$ terms yields $N^{-1} \sum_{m=0}^{\infty} a^{2m+1} = a/N(1 - a^2)$, and we have

$$\operatorname{Re} \sum_{n=0}^{N-1} \frac{1}{1 - a e^{i\theta_n}} = \frac{1}{1 - a^{2N}} + \frac{a}{N(1 - a^2)}. \quad (70)$$

Equating (69) with (70) we obtain

$$S_1(0) = \frac{1}{1 - a^2} \left[\left(\frac{1 + a^{2N}}{1 - a^{2N}} \right) + \frac{2a}{N(1 - a^2)} \right]. \quad (71)$$

To evaluate $S_1(\ell)$ for general ℓ , we consider the summation

$$\begin{aligned} \operatorname{Re} \frac{1}{N} \sum_{n=0}^{N-1} \frac{1 - (a e^{i\theta_n})^\ell}{1 - a e^{i\theta_n}} &= \operatorname{Re} \frac{1}{N} \sum_{n=0}^{N-1} \frac{(1 - a^\ell e^{i\ell\theta_n})(1 - a e^{-i\theta_n})}{|1 - a e^{i\theta_n}|^2} \\ &= S_1(0) - a S_1(1) - a^\ell S_1(\ell) + a^{\ell+1} S_1(\ell - 1) \end{aligned} \quad (72)$$

where the second line is obtained by writing out the real part of the summand as in (69). On the other hand, by expanding the summand we have

$$\begin{aligned} \operatorname{Re} \frac{1}{N} \sum_{n=0}^{N-1} \frac{1 - (a e^{i\theta_n})^\ell}{1 - a e^{i\theta_n}} &= \operatorname{Re} \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{\ell-1} a^m \exp(i\pi mn/N) \\ &= 1 + \operatorname{Re} \frac{1}{N} \sum_{m=1}^{\ell-1} a^m \left(\frac{1 - (-1)^m}{1 - \exp(i\pi m/N)} \right) \\ &= 1 + \frac{a(1 - a^\ell)}{N(1 - a^2)} \quad \ell = \text{even} < 2N \\ &= 1 + \frac{a(1 - a^{\ell-1})}{N(1 - a^2)} \quad \ell = \text{odd} < 2N \end{aligned} \quad (73)$$

where again we have used (25).

Equating (73) with (72) and using (68) and (71), we obtain the recursion relation

$$S_N(\ell) - a S_N(\ell - 1) = A a^{-\ell} + B_\ell \quad (74)$$

where

$$A = \frac{a^{2N}}{1 - a^{2N}} \quad B_\ell = \frac{a^{(1+(-1)^\ell)/2}}{N(1 - a^2)}. \quad (75)$$

The recursion relation (74) can be solved by standard means. Define the generating function

$$G_\alpha(t) = \sum_{\ell=0}^{\infty} S_\alpha(\ell) t^\ell \quad \alpha = 1, 2. \quad (76)$$

Multiplying (74) by t^ℓ and sum over ℓ , we obtain

$$(1 - at)G_1(t) - S_1(0) = \frac{Aa^{-1}t}{1 - a^{-1}t} + \frac{t + at^2}{N(1 - a^2)(1 - t^2)}. \tag{77}$$

This leads to

$$\begin{aligned} G_1(t) &= \frac{1}{1 - at} \left[S_1(0) + \frac{Aa^{-1}t}{1 - a^{-1}t} + \frac{t + at^2}{N(1 - a^2)(1 - t^2)} \right] \\ &= \frac{1}{(1 - a^2)(1 - a^{2N})} \left[\frac{1}{1 - at} + \frac{a^{2N}}{1 - a^{-1}t} \right] \\ &\quad + \frac{1}{2N(1 - a)^2(1 - t)} - \frac{1}{2N(1 + a)^2(1 + t)} \end{aligned}$$

from which one obtains

$$\begin{aligned} S_1(\ell) &= \frac{a^\ell + a^{2N-\ell}}{(1 - a^2)(1 - a^{2N})} + \frac{1}{2N(1 - a)^2} - \frac{(-1)^\ell}{2N(1 + a)^2} \\ &= \frac{a^\ell + a^{2N-\ell}}{(1 - a^2)(1 - a^{2N})} + \frac{1}{2N} \left[\frac{4a}{(1 - a^2)^2} + \frac{1 - (-1)^\ell}{(1 + a^2)^2} \right]. \end{aligned} \tag{78}$$

It follows that using $I_1(\ell) = 2aS_1(\ell)$ we obtain (61) after setting $a = e^{-|\lambda|}$. □

Proof of (62). Again, we first evaluate $S_2(0)$ by carrying out the summation

$$\operatorname{Re} \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{1 - a e^{i2\theta_n}} \quad a < 1 \tag{79}$$

in two different ways. First as in (69) we have

$$\operatorname{Re} \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{1 - a e^{i2\theta_n}} = \frac{1}{2} [1 + (1 - a^2)S_2(0)] \tag{80}$$

where $S_2(\ell)$ is defined in (66). Secondly, by expanding the summand we have

$$\frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{1 - a e^{i2\theta_n}} = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{\ell=0}^{\infty} a^\ell \exp(i2\ell n\pi/N) = \frac{1}{1 - a^N} \tag{81}$$

where by carrying out the summation over n for fixed ℓ all terms in (70) vanish except those with $\ell = mN, m = 0, 1, 2, \dots$. Equating (81) with (80) we obtain

$$S_2(0) = \frac{1}{1 - a^2} \left(\frac{1 + a^N}{1 - a^N} \right) \tag{82}$$

and from (68)

$$S_2(1) = \frac{1}{1 - a^N}.$$

We consider next the summation

$$\operatorname{Re} \frac{1}{N} \sum_{n=0}^{N-1} \frac{1 - (a e^{i2\theta_n})^\ell}{1 - a e^{i2\theta_n}} \quad a < 1. \tag{83}$$

Evaluating the real part of the summand directly as in (72), we obtain

$$\operatorname{Re} \frac{1}{N} \sum_{n=0}^{N-1} \frac{1 - (a e^{i2\theta_n})^\ell}{1 - a e^{i2\theta_n}} = S_2(0) - aS_2(1) - a^\ell S_2(\ell) + a^{\ell+1} S_2(\ell - 1). \tag{84}$$

Secondly, expanding the summand in (83) we obtain

$$\begin{aligned} \frac{1}{N} \sum_{n=0}^{N-1} \frac{1 - (a \exp(i2\theta_n))^\ell}{1 - a \exp(i2\theta_n)} &= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{\ell-1} a^m \exp(i2\pi mn/N) \\ &= \frac{1}{N} \left[N + \sum_{m=1}^{\ell-1} \frac{1 - \exp(i2m\pi)}{1 - \exp(i2m\pi/N)} \right] \\ &= 1 \quad m < \ell \leq N. \end{aligned} \quad (85)$$

Equating (85) and (84) and making use of (82) for $S_2(0)$, we obtain

$$S_2(\ell) - aS_2(\ell - 1) = \frac{a^{N-\ell}}{1 - a^N}. \quad (86)$$

The recursion relation (86) can be solved as in the above. Define the generating function $G_2(t)$ by (76). We find

$$\begin{aligned} G_2(t) &= \frac{1}{1 - at} \left[S_2(0) + \frac{a^{N-1}t}{(1 - a^N)(1 - a^{-1}t)} \right] \\ &= \frac{1}{(1 - a^2)(1 - a^{2N})} \left[\frac{1}{1 - at} + \frac{a^N}{1 - a^{-1}t} \right] \end{aligned} \quad (87)$$

from which one reads off

$$S_2(\ell) = \frac{a^\ell + a^{N-\ell}}{(1 - a^2)(1 - a^{2N})}. \quad (88)$$

Using the relation $I_2(\ell) = 2aS_2(\ell)$ with $a = e^{-|\lambda|}$, we obtain (62). \square

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